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Probabilistic approach to the Lamé equations of linear elastostatics

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Abstract

A probabilistic approach to systems of partial differential equations is developed on the basis of the well-known Feynman–Kac and Bismut formulas providing explicit probabilistic representations of the solutions and of their derivatives of scalar differential equations. Some numerical examples are also included. In particular the Lamé equations of elastostatics are solved and the results are compared with some known exact analytic solutions to demonstrate the efficiency of the approach.

Keywords: Lamé equations; Feynman–Kac–Bismut formula; Brownian motion

1. Introduction

It is attractive to have solutions of applied mathematical problems in forms that permit the computations of interest at isolated points without computing the functions on massive meshes. For many problems described by partial differential equations such solutions are delivered by the so-called Feynman–Kac formulas (Dynkin, 1965; Freidlin, 1985; Simon, 1979) involving mathematical expectations of specified functionals on random walks associated with the equations under consideration. These formulas and their numerous variations may be considered from many different points of view, some of which are discussed in (Simon, 1979). Since all of the approaches somehow involve the averaging over trajectories of random walks, it is appropriate to use the term 'random walk method' for any method of analysis in which the formulas require averaging over random motions.

Random walk methods have been used since the 1920s (Courant et al., 1928; Khinchin, 1933; Petrovsky, 1934; Wiener, 1923) for the analysis of scalar parabolic and elliptic positive-definite partial differential equations describing, respectively, diffusion processes and equilibrium states. The advantages of these methods include versatility, unrestrictive requirements on the problem's data, the possibility of computing functions of interest at isolated points, and the possibility of implementations employing simple scalable algorithms with virtually unlimited capability for parallel processing. Nevertheless, the practical impact of

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random walk methods has been limited, partially because these methods have not been extended to problems of wave propagation or to problems described by systems of coupled partial differential equations, such as the Lamé equations of the theory of elasticity or the Maxwell equations of electrodynamics. However, in recent years there has been increased activity in the further development of probabilistic methods of analysis of physical problems. Thus, in (Busnello, 1999) the random walk method is applied to the analysis of the Navier–Stokes equations of hydrodynamics, and in (Chati et al., 2001; Grigoriu, 2000; Shia and Hui, 2000) random walks are applied to certain restricted problems of elasticity. Quite recently attempts have also been made to apply similar methods to steady flow computations (Hunt et al., 1995) as well as to acoustics and electromagnetics (Galdi et al., 2000, 2001, September–October; Nevels et al., 2000; Schlottmann, 1999), and these methods are increasingly used for analysis of geophysical wave propagation such as in (Bal et al., 1999, 2000), where the competitiveness of random walk methods in wave propagation is discussed.

Following this trend we recently launched a program aimed at developing simple, but theoretically exact, random walk solutions of problems of elastic wave propagation. In the first stage, the solutions of the scalar Helmholtz equation in the entire space and in simple exterior domains were represented in the probabilistic form (Budaev and Bogy, 2001, 2002a,b) which was numerically tested on elementary examples admitting simple analytic solutions for comparison. Later, in (Budaev and Bogy, 2002a,b,c), it was shown that the random walk method makes it possible to describe such phenomena of wave propagation as backscattering, which is predicted neither by the ray method nor by a more general method of parabolic equations (Fock and Leontovich, 1946; Fock, 1965). Most recently in (Budaev and Bogy, 2003a,b,c) we applied the method to the analysis of wave propagation in canonical domains: waves in wedges and cones are considered in (Budaev and Bogy, 2003b); waves in exterior cylindrical and spherical domains are discussed in (Budaev and Bogy, 2003c); and in (Budaev and Bogy, 2003a) the important problem of diffraction by a plane wedge-shaped screen is solved by the method from (Budaev and Bogy, 2003a,b,c).

Here we present an extension of the random walk method to a class of systems of linear second-order partial differential equations that includes the Lamé equations of the theory of elasticity. For the sake of a transparent presentation of the basic ideas this paper is restricted to the discussion of the Lamé equations in the entire isotropic homogeneous space, or, more generally, to the discussion of systems of coupled differential equations with constant coefficients in the entire space. Certainly, the analysis of such equations from another, even non-conventional, point of view may not create new knowledge about the problems whose closed-form analytic solutions are easily available. We fully appreciate this point, but note that without demonstrating the success on elementary models it is hard to justify further efforts in the development of a new method. In future papers we plan to consider equations of elastostatics in bounded, not necessarily homogeneous, domains with various boundary conditions, and to address dynamic problems of propagation and scattering of elastic waves.

In this paper we start from a brief and elementary discussion of the Feynman–Kac formulas, which represent solutions of scalar partial differential equations, and of the Bismut formulas representing derivatives of those solutions. Although these results are rather standard, they are included here to make the present developments more accessible to a broader audience, which includes specialists working with applications who may not be familiar with advanced stochastic calculus. Then, combining in a certain order the results for scalar equations we obtain the probabilistic solutions of systems of second-order differential equations.

2. Feynman–Kac formulas

The probabilistic approach to partial differential equations has been known since the 1920s but it became widely known several years later, after the papers of Feynman (1942, 1948) and Kac (1949, 1951) where the solutions of the Schrödinger and the diffusion equations were represented in the form which is now usually referred to as the Feynman–Kac formulas. Since then, Feynman–Kac formulas have been derived using several distinctively different methods as discussed, for example, in (Simon, 1979). One such method is outlined here in order to make the paper self-contained and to simplify the following discussion of the less known material presented in Section 3.

Consider the initial value problem

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\sum_{i=1}^{N} \Lambda_{ii}^2 \frac{\partial^2 \phi}{\partial x_i^2} + \vec{A}\cdot\vec{\nabla}\phi + B\phi, \quad \phi(x,0) = F(x), \tag{2.1}$$

whose coefficients Λ_{ii}^2, $\vec{A}$ and B are smooth functions of the spatial variable $x \in \mathbb{R}^N$.

If the coefficients Λ_{ii} and B are constants, and $\Lambda_{ii} \neq 0$, then the solution $\phi(x,t)$ of the problem (2.1) is given by either of the two following integrals:

$$\phi(x,t) = e^{Bt}\int_{\mathbb{R}^N} \frac{e^{-[\Lambda^{-1}(y-x-t\vec{A})]^2/2t}}{(2\pi t)^{N/2}\Lambda_{11}\Lambda_{22}\cdots\Lambda_{NN}} F(y)\,dy \tag{2.2}$$

$$= e^{Bt}\int_{\mathbb{R}^N} \frac{e^{-\vec{w}^2/2t}}{(2\pi t)^{N/2}} F(x + t\vec{A} + \Lambda\vec{w})\,d\vec{w}, \tag{2.3}$$

where Λ is the diagonal matrix

$$\Lambda = \mathrm{diag}[\Lambda_{11}, \Lambda_{22}, \ldots, \Lambda_{NN}], \tag{2.4}$$

generated by the coefficients of (2.1). If the coefficients of (2.1) are not constants, then the integrals (2.2) and (2.3) do not solve (2.1), but, nevertheless, they can be used to construct an explicit probabilistic solution of (2.1).

To solve (2.1) with non-constant coefficients we observe that the evolution equation (2.1) generates a family of time translation operators

$$\mathfrak{T}_t : \phi(x, t_0) \to \phi(x, t_0 + t), \tag{2.5}$$

transferring the solution of (2.1) at the instant t_0 to the solution at the instants $t_0 + t$. Then the solution $\phi(x,t)$ of (2.1) can be represented as a composition

$$\phi(x,t) = \underbrace{\mathfrak{T}_\varepsilon \cdot \mathfrak{T}_\varepsilon \cdots \mathfrak{T}_\varepsilon}_{n\text{-times}} F(x), \quad \varepsilon = \frac{t}{n}, \tag{2.6}$$

of consecutive time transitions on a time interval ε. If $\varepsilon \ll 1$ then the operators $\mathfrak{T}_\varepsilon$ can be approximated by the integrals (2.2) or (2.3) with the kernel corresponding to the matrix Λ from (2.4) frozen at the point x. Then, passing to the limit $t \to 0$ we arrive at the solution of (2.1) in the form

$$\phi(x,t) = \lim_{\varepsilon\to 0} \phi_\varepsilon(x,t), \quad n\varepsilon = t, \tag{2.7}$$

with the approximation $\phi_\varepsilon(x,t)$ determined by either of the formulas

$$\phi_\varepsilon(x,t) = e^{\varepsilon B(x)}\int_{\mathbb{R}^N} \frac{e^{-[\Lambda^{-1}(x)(y-x-\varepsilon\vec{A}(x))]^2/2\varepsilon}}{(2\pi t)^{N/2}[\sigma_1(x)\sigma_2(x)\cdots\sigma_N(x)]} \phi_\varepsilon(y, t-\varepsilon)\,dy \tag{2.8}$$

$$= e^{\varepsilon B(x)}\int_{\mathbb{R}^N} \frac{e^{-\vec{w}^2/2\varepsilon}}{(2\pi t)^{N/2}} \phi_\varepsilon(x + \varepsilon\vec{A}(x) + \Lambda(x)\vec{w}, t-\varepsilon)\,d\vec{w}, \tag{2.9}$$

which should be applied recursively exactly n times, until we arrive at the integral with the pre-defined function $\phi_\varepsilon(y,0) \equiv F(y)$ in the right-hand side.

The evaluation of (2.9) is based on the Monte-Carlo quadrature formula

$$\int_{\mathbb{R}^N} f(\vec{w}) \frac{e^{-\vec{w}^2/2D}}{\sqrt{2\pi D}} \, d\vec{w} = \mathbf{E}\{f(\Delta\vec{w})\}, \tag{2.10}$$

where $\mathbf{E}$ denotes the mathematical expectation computed over the random vector $\Delta\vec{w} \in \mathbb{R}^N$ normally distributed with the dispersion D. Then, applying (2.10) to (2.9) we express $\phi_\varepsilon(x,t)$ as the mathematical expectation

$$\phi_\varepsilon(x,t) = \mathbf{E}\left\{\phi_\varepsilon\left(x + \varepsilon\vec{A}(x) + \Lambda(x)\Delta\vec{w}, t - \varepsilon\right)e^{\varepsilon B(x)}\right\}, \tag{2.11}$$

with the averaging over the random vectors $\Delta\vec{w} \in \mathbb{R}^N$ normally distributed with the standard deviation $D = \varepsilon$. Recursively applying this formula n times we arrive at the representation

$$\phi_\varepsilon(x,t) = \mathbf{E}\left\{F(\xi_n)e^{\varepsilon[B(\xi_0)+B(\xi_1)+\cdots+B(\xi_{n-1})]}\right\}, \quad n\varepsilon = t, \tag{2.12}$$

where the mathematical expectation is computed over the n-legged discrete trajectories

$$x \equiv \xi_0 \to \xi_1 \to \cdots \to \xi_n, \quad \xi_{k+1} = \xi_k + \varepsilon\vec{A}(\xi_n) + \Lambda(\xi_n)\Delta\vec{w}_{k+1}, \tag{2.13}$$

determined by the independent random vectors $\Delta\vec{w}_1, \Delta\vec{w}_2, \ldots, \Delta\vec{w}_n$, distributed according to the normal law with the standard deviation $D = \varepsilon$. Finally, passing in (2.12) to the limit $\varepsilon \to 0$ we obtain the well-known Feynman–Kac formula (Dynkin, 1965; Freidlin, 1985)

$$\phi(x,t) = \mathbf{E}\left\{F(\xi_t^x)e^{\int_0^t B(\xi_s^x)\,ds}\right\}, \tag{2.14}$$

where the averaging is extended over the trajectories of the continuous random motion ξ_t^x governed by Ito's stochastic equation

$$d\xi_t^x = \vec{A}(\xi_t^x)dt + \Lambda(\xi_t^x)d\vec{w}_t, \quad \xi_0^x = x, \tag{2.15}$$

with $\vec{w}_t$ denoting the standard N-dimensional Brownian motion (Wiener process).

It should be emphasized that the Feynman–Kac formula (2.14) that provides the solution of the Cauchy problem (2.1) in the entire space $\mathbb{R}^N$ is so versatile that it can be used in several different ways to derive probabilistic solutions of problems like (2.1) formulated in a domain G with conditions imposed on its boundary ∂G. Thus, the Dirichlet problem on the domain $G \subset \mathbb{R}^N$ consisting of the equation and the initial condition from (2.1), and the boundary condition

$$\phi(x,t)|_{\partial G} = 0, \tag{2.16}$$

can be treated as a problem (2.1) in the entire space $\mathbb{R}^N$ with the coefficient $B(x)$ extended outside of G as

$$B(x) = -\infty, \quad x \notin G, \tag{2.17}$$

and with the coefficients $\Lambda_{kk}(x)$, $\vec{A}(x)$ extended outside G as arbitrary bounded functions. Then, applying (2.14) and taking into account (2.17) we can readily show that the solution of the problem (2.1) and (2.16) is given by the mathematical expectation

$$\phi(x,t) = \mathbf{E}\left\{\chi(\tau - t)F(\xi_t^x)e^{\int_0^t B(\xi_s^x)ds}\right\}, \quad \chi(\tau - t) = \begin{cases} 1, & \text{if } t < \tau, \\ 0, & \text{if } t \geqslant \tau, \end{cases} \tag{2.18}$$

where τ is the exit time, defined as the instant when the trajectory ξ_t^x touches the boundary ∂G for the first time.

Another approach to the probabilistic solution of the Dirichlet problem (2.1) and (2.16) is to extend the coefficients and the initial data of the problem (2.1) outside of G by the formulas

$$\Lambda_{kk}(x) = \vec{A}(x) = F(x) = 0, \quad B(x) = -1 \qquad \text{for } x \notin G. \tag{2.19}$$

Then, applying (2.14) and taking into account (2.19) we again arrive at (2.18).

Finally, to demonstrate the versatility of the random walks approach we outline one more method leading to explicit probabilistic solutions of the Dirichlet problem (2.1) and (2.16) in a form distinctively different from (2.18).

Let us restrict ourselves, for simplicity, to the heat-conduction problem

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\nabla^2 \phi + B\phi, \quad \phi(x,0) = F(x), \quad \phi(x,t)|_{x\in\partial G} = 0, \qquad B < 0, \tag{2.20}$$

which is a particular but representative case of (2.1) and (2.16), and seek its solution in the product form

$$\phi(x,t) = g(x)\psi(x,t), \tag{2.21}$$

where $\psi(x,t)$ is a new unknown function and $g(x)$ is an auxiliary smooth function on G satisfying the conditions

$$g(x) = 0, \quad 0 < \|\vec{n}(x)\cdot\vec{\nabla}g(x)\| < \text{const} \quad \text{for } x \in \partial G, \tag{2.22}$$

$$g(x) > 0, \quad \nabla^2 g(x) < 0 \quad \text{for } x \in G, \tag{2.23}$$

where $\vec{n}(x)$ is the unit vector normal to ∂G at the point x and oriented inward to G. Such a function always exists and can usually be defined by simple elementary formulas, utilizing specifics of the domain G. For example, if G is the ball $\|x\| < 1$, then $g(x)$ can be defined as $g(x) = 1 - \|x\|^2$. In general, $g(x)$ can always be defined as a solution of the Dirichlet problem $\nabla^2 g = -1$, $g|_{\partial G} = 0$.

Substitution of (2.21) into (2.20) results in the problem for ψ

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\nabla^2\psi + \frac{\vec{\nabla}g}{g}\cdot\vec{\nabla}\psi + \left(B + \frac{\nabla^2 g}{2g}\right)\psi, \quad \psi(x,0) = \frac{F(x)}{g(x)}, \tag{2.24}$$

matching the structure (2.1), so that its solution is provided by the Feynman–Kac formula

$$\psi(x,t) = \mathbf{E}\left\{\frac{F(\xi_t^x)}{g(\xi_t^x)} e^{\int_0^t \widetilde{B}(\xi_s^x)\,ds}\right\}, \quad \widetilde{B}(\xi) = B(\xi) + \frac{\nabla^2 g(\xi)}{2g(\xi)} < 0, \tag{2.25}$$

with averaging over the trajectories of the random motion ξ_t^x governed by the equation

$$d\xi_t^x = d\vec{w}_t + \vec{A}dt, \quad \vec{A} = \vec{\nabla}g/g, \tag{2.26}$$

driven by the standard Brownian motion $\vec{w}_t$. Due to the inequalities (2.23), the drift $\vec{A}$ from (2.26) is oriented from the boundary ∂G inwards to G. So, the closer ξ_t^x comes to ∂G the more deterministically ξ_t^x is pulled back into G. As a result the trajectory of ξ_t^x stays inside G, softly bouncing from the unreachable boundary ∂G. Since ξ_t^x never leaves G, the expression (2.25) involves values of $F(x)$ only inside G where they are defined by the problem (2.24). Therefore, $\psi(x,t)$ is consistently defined by (2.25) anywhere inside G and it has finite values on ∂G, which guarantees that the product (2.21) obeys all the conditions of the Cauchy problem (2.24) with Dirichlet boundary conditions.

3. Bismut formulas

Since the Feynman–Kac formula (2.14) provides the solution $\phi(x,t)$ of the problem (2.1) at any point $x \in \mathbb{R}^N$, it also formally determines the spatial derivatives of $\phi(x,t)$, as well. However, (2.14) does not suggest a direct method for computing such derivatives, which severely restricts the use of probabilistic methods in applied problems, where it is often equally as important to compute the gradient of the unknown function as the function itself. This chronic drawback of the original Feynman–Kac formulas has been eliminated by the so-called Bismut formula (Bismut, 1984), originally published in 1984 and then widely discussed (Driver, 1997; Elworthy, 1992; Elworthy and Li, 1994; Norris, 1993) in the 1990s. To gain some insight into the nature of the Bismut-like formulas we derive here some such formulas representing derivatives of the solution of the problem (2.1) with constant coefficients.

We first derive, for example, an expression for the second-order derivative

$$D^2_{\vec{v}_1\vec{v}_2}\phi(x,t) = \left.\frac{\partial^2\phi(x+\lambda_1\vec{v}_1+\lambda_2\vec{v}_2,t)}{\partial\lambda_1\partial\lambda_2}\right|_{\lambda_1=0,\,\lambda_2=D}, \tag{3.1}$$

of the solution of (2.1) along the independent vectors $\vec{v}_1$ and $\vec{v}_2$.

If the coefficients of the problem (2.1) are constants, then the integral (2.9) representing its approximate solution $\phi_\varepsilon(x,t)$ through $\phi_\varepsilon(x,t-\varepsilon)$ degenerates to the form

$$\phi_\varepsilon(x,t) = \frac{e^{\varepsilon B}}{(2\pi\varepsilon)^{N/2}}\int_{\mathbb{R}^N} e^{-\vec{w}^2/2\varepsilon}\phi_\varepsilon(x+\varepsilon\vec{A}+\Lambda\vec{w},t-\varepsilon)\,d\vec{w}, \tag{3.2}$$

which eventually leads, after n iterations, to the expression

$$\phi_\varepsilon(x,t) = \frac{e^{n\varepsilon B}}{\left(\sqrt{2\pi\varepsilon}\right)^{nN}}\int_{\mathbb{R}^{nN}} e^{-(\vec{w}_1^2+\cdots+\vec{w}_n^2)/2\varepsilon}\phi_\varepsilon(x+\xi_n,0)\,d\vec{w}_1\ldots d\vec{w}_n, \quad n\varepsilon = t, \tag{3.3}$$

where

$$\xi_n = n\varepsilon\vec{A} + \Lambda(\vec{w}_1+\vec{w}_2+\cdots+\vec{w}_n), \quad n\varepsilon = t, \tag{3.4}$$

and

$$\phi_\varepsilon(x+\xi_n,0) \equiv F(x+\xi_n), \tag{3.5}$$

in which $F(x)$ is the initial-data function from (2.1). Therefore, combining (2.7) with (3.3)–(3.5) we conclude that the derivative $D^2_{\vec{v}_1\vec{v}_2}\phi$ can be represented as the limit

$$D^2_{\vec{v}_1\vec{v}_2}\phi(x,t) = \lim_{\varepsilon\to 0} D^2_{\vec{v}_1\vec{v}_2}\phi_\varepsilon(x,t), \tag{3.6}$$

of the approximation

$$D^2_{\vec{v}_1\vec{v}_2}\phi_\varepsilon(x,t) = \left.\frac{d^2\phi_\varepsilon(x+\lambda_1\vec{v}_1+\lambda_2\vec{v}_2,t)}{d\lambda,d\lambda_2}\right|_{\lambda_1=0,\,\lambda_2=0}, \tag{3.7}$$

where

$$\phi_\varepsilon(x+\lambda_1\vec{v}_1+\lambda_2\vec{v}_2,t) = \frac{e^{n\varepsilon B}}{\left(\sqrt{2\pi\varepsilon}\right)^{nN}}\int_{\mathbb{R}^{nN}} e^{-(\vec{w}_1^2+\cdots+\vec{w}_n^2)/2\varepsilon}F(y_n)d\vec{w}_1,\ldots,d\vec{w}_n, \tag{3.8}$$

and

$$y_n = x+\lambda_1\vec{v}_1+\lambda_2\vec{v}_2+\Lambda[\vec{w}_1+\vec{w}_2+\cdots+\vec{w}_n]+n\varepsilon\vec{A}, \quad n\varepsilon = t. \tag{3.9}$$

Let us select two independent indices i and j from the range $1\leqslant i,j\leqslant n$, and re-arrange the order of the terms in (3.8) representing y_n as

$$y_n = x+t\vec{A}+\Lambda[\vec{w}_1+\cdots+(\vec{w}_i+\lambda_1\Lambda^{-1}\vec{v}_1)+\cdots+(\vec{w}_j+\lambda_2\Lambda^{-1}\vec{v}_2)+\cdots+\vec{w}_n], \tag{3.10}$$

where the indices i and j are not required to follow in the shown order $i < j$. Then, the integral (3.8) is reduced by the substitutions

$$\vec{w}_i + \lambda_1 \Lambda^{-1}\vec{v}_1 \to \vec{w}_i, \quad \vec{w}_j + \lambda_2 \Lambda^{-1}\vec{v}_2 \to \vec{w}_j, \tag{3.11}$$

to the form

$$\phi_\varepsilon(x + \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2, t) = \frac{tB}{\left(\sqrt{2\pi\varepsilon}\right)^{nN}} \int_{\mathbb{R}^{nN}} \mathrm{e}^{-\Phi^{i,j}_{\lambda_1\lambda_2}(\vec{w}_1,\ldots,\vec{w}_n)/2\varepsilon} F(z_n)\, \mathrm{d}\vec{w}_1, \ldots, \mathrm{d}\vec{w}_n, \tag{3.12}$$

where

$$\Phi^{i,j}_{\lambda_1\lambda_2}(\vec{w}_1, \ldots, \vec{w}_n) = \vec{w}_1^2 + \cdots + (\vec{w}_i - \lambda_1\Lambda^{-1}\vec{v}_1)^2 + \cdots + (\vec{w}_j - \lambda_2\Lambda^{-1}\vec{v}_2)^2 + \cdots + \vec{w}_n^2 \tag{3.13}$$

and the points

$$z_n = x + t\vec{A} + \Lambda\vec{w}_1 + \Lambda\vec{w}_2 + \cdots + \Lambda\vec{w}_n, \tag{3.14}$$

do not depend on λ_1, λ_2. Then, straightforward differentiation of (3.12) by λ_1 and λ_2 yields

$$D^2_{\vec{v}_1\vec{v}_2}\phi_\varepsilon(x,t) = \frac{n^2 \mathrm{e}^{tB}}{t^2} \int_{\mathbb{R}^{nN}} F(z_n)\left(\Lambda^{-1}\vec{v}_1 \cdot \vec{w}_i\right)\left(\Lambda^{-1}\vec{v}_2 \cdot \vec{w}_j\right) \frac{\mathrm{e}^{-(\vec{w}_1^2+\cdots+\vec{w}_n^2)/2\varepsilon}}{\left(\sqrt{2\pi\varepsilon}\right)^{nN}}\, \mathrm{d}\vec{w}_1, \ldots, \mathrm{d}\vec{w}_n, \tag{3.15}$$

and evaluating the integrals with respect to $\vec{w}_k$ by the quadratures (2.10) we obtain the formula

$$D^2_{\vec{v}_1\vec{v}_2}\phi_\varepsilon(x,t) = \frac{n^2 \mathrm{e}^{tB}}{t^2} \mathbf{E}\left\{F(\xi_n)\left(\Lambda^{-1}\vec{v}_1 \cdot \Delta\vec{w}_i\right)\left(\Lambda^{-1}\vec{v}_2 \cdot \Delta\vec{w}_j\right)\right\}, \tag{3.16}$$

which represents the second-order derivatives of $\phi_\varepsilon(x,t)$ as a weighted average of the values of the function $F(x)$ at the end-points

$$\xi_n = x + t\vec{A} + \Lambda(\Delta\vec{w}_1 + \Delta\vec{w}_2 + \cdots + \Delta\vec{w}_n), \tag{3.17}$$

of the discrete Brownian trajectories determined by the independent random vectors $\Delta\vec{w}_1, \Delta\vec{w}_2, \ldots, \Delta\vec{w}_n$, distributed by the Gaussian law with the standard deviation $D = \varepsilon \equiv t/n$.

Although the formulas (3.16) explicitly represent the derivatives of the approximate solution ϕ_ε of the problem (2.1), the presence of the factor n^2 makes these formulas unsuitable for passing to the limit $t/n \equiv \varepsilon \to 0$. However, since the representations (3.16) remain valid for any indices i and j from the range $1 \leqslant i, j \leqslant n$, it is possible to combine these formulas into expressions that do admit passage to the limit $\varepsilon \to 0$. Thus, the arithmetic mean value of all the formulas (3.16) has the structure

$$D^2_{\vec{v}_1\vec{v}_2}\phi_\varepsilon(x,t) = \frac{1}{t^2}\mathbf{E}\left\{F(\xi_n)\mathrm{e}^{tB}\sum_{i,j=1}^{n}(\Lambda^{-1}\vec{v}_1 \cdot \Delta\vec{w}_i)(\Lambda^{-1}\vec{v}_2 \cdot \Delta\vec{w}_j)\right\}, \tag{3.18}$$

which converges in the limit $n \to \infty$ to the Bismut formula (Elworthy and Li, 1994)

$$D^2_{\vec{v}_1\vec{v}_2}\phi(x,t) = \mathbf{E}\left\{F(\xi^x_t)\mathrm{e}^{tB}\mathscr{D}^2_{\vec{v}_1\vec{v}_2}([\xi^x_t])\right\}, \tag{3.19}$$

where

$$\mathscr{D}^2_{\vec{v}_1\vec{v}_2}([\xi^x_t]) = \frac{1}{t^2}\int_0^t \int_0^t \left(\Lambda^{-1}\vec{v}_1 \cdot \mathrm{d}\vec{w}_{s_1}\right)\left(\Lambda^{-1}\vec{v}_2 \cdot \mathrm{d}\vec{w}_{s_2}\right), \tag{3.20}$$

is a functional depending on the trajectory of the random motion

$$\xi^x_t = x + t\vec{A} + \vec{w}_t, \tag{3.21}$$

driven by the standard Brownian motion $\vec{w}_t$.

It is instructive to observe that the formula (3.19) is not unique, but is a very particular representative of a broad family of similar formulas that can be derived from the basic formulas (3.16) by different kinds of averaging with respect to the indices i and j. For instance, any function $p_t(s_1, s_2)$, integrable on the square $0 \leqslant s_1,\ s_2 \leqslant t$ generates the representation (3.19) with the functional

$$\mathscr{D}^2_{\vec{v}_1\vec{v}_2}([\xi^x_t]) = \frac{1}{\mathscr{P}_t}\int_0^t\int_0^t p_t(s_1,s_2)\left(\Lambda^{-1}\vec{v}_1\cdot \mathrm{d}\vec{w}_{s_1}\right)\left(\Lambda^{-1}\vec{v}_1\cdot \mathrm{d}\vec{w}_{s_2}\right), \tag{3.22}$$

where

$$\mathscr{P}_t = \int_0^t\int_0^t p_t(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2. \tag{3.23}$$

Indeed, to derive (3.19) and (3.22) it suffices to average formulas (3.16) with the weights

$$p^n_{i,j} = \int_{t_i-1}^{t_i}\mathrm{d}s_1\int_{t_j-1}^{t_j} p_t(s_1,s_2)\,\mathrm{d}s_2, \qquad t_i = i\varepsilon, \quad t_j = j\varepsilon, \tag{3.24}$$

and then to pass to the limit $\varepsilon \to 0$.

It should also be mentioned that formulas (3.19) and (3.22) admit straightforward generalizations providing probabilistic representations of any derivatives of the solution $\phi(x,t)$ of the problem (2.1). Thus, the derivatives of the kth order along the k-tuple $\mathfrak{v} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of independent vectors $\vec{v}_i$, are represented by the mathematical expectations

$$D^k_{\mathfrak{v}}(x,t) = \mathbf{E}\left\{F(\xi^x_t)\mathrm{e}^{tB}\mathscr{D}^k_{\mathfrak{v}}([\xi^x_t])\right\}, \quad \mathfrak{v} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}, \tag{3.25}$$

where the 'differentiating' functionals $\mathscr{D}^k_{\mathfrak{v}}([\xi^x_t])$ are determined on the trajectories of the random motion ξ^x_t from (3.21) by the integrals

$$\mathscr{D}^k_{\mathfrak{v}}([\xi^x_t]) = \frac{1}{\mathscr{P}_t}\int_0^t\int_0^t\cdots\int_0^t p_t(s_1,s_2,\ldots,s_k)\prod_{i=1}^k\left(\Lambda^{-1}\vec{v}_i\cdot \mathrm{d}\vec{w}_{s_i}\right), \tag{3.26}$$

$$\mathscr{P}_t = \int_0^t\int_0^t\cdots\int_0^t p_t(s_1,s_2,\ldots,s_k)\,\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_k, \tag{3.27}$$

involving an arbitrary density $p(s_1,s_2,\ldots,s_k)$ integrable in the domain $0 < s_1, s_2, \ldots, s_k < t$.

In conclusion of this brief discussion of the Bismut formulas it must be emphasized that they admit extensions (Bismut, 1984; Driver, 1997; Elworthy, 1992; Elworthy and Li, 1994; Norris, 1993) going far beyond the elementary cases considered here, and they deliver, in particular, probabilistic expressions for the derivatives of the solutions of the problem (2.1) with non-constant coefficients, which makes it possible to apply the techniques outlined at the end of Section 2 and to obtain probabilistic expressions for the derivatives of the solutions of the problem (2.1) in a finite domain with imposed boundary conditions.

4. Probabilistic formulas for elliptic equations

Here we consider a second-order elliptic equation

$$\frac{1}{2}\sum_{i=1}^N \Lambda^2_{ii}\frac{\partial^2 u}{\partial x_i^2} + \vec{A}\cdot\vec{\nabla}u + Bu + F = 0, \tag{4.1}$$

with constant coefficients Λ_{ii}, $\vec{A}$ and B.

It is clear that (4.1) can be obtained by the integration of the parabolic equation (2.1) by the time variable from zero to infinity. From this it follows that if the integral

$$u(x) = \int_0^\infty \phi(x,t)\,\mathrm{d}t, \tag{4.2}$$

where $\phi(x,t)$ is the solution of (2.1), converges then it represents the solution $u(x)$ of the elliptic equation (4.1). As for the convergence of (4.2) it may be guaranteed by different conditions on the coefficients of Eq. (4.1). For example, the requirement $B<0$ guarantees the convergence of (4.2) to the solution of (4.1) with any bounded function $F(x)$ and with any vector coefficient $\vec{A}$. Also, the inequality $\vec{A}\neq 0$ and the estimate $|F(x)| < O(\mathrm{e}^{-|x|^\alpha})$, $\alpha>1$, guarantee the convergence of (4.2) independently of the value of the coefficient B.

Assume that the integral (4.2) corresponding to Eq. (4.1) converges. Then, combining (4.2) with the Feynman–Kac formula (2.14) we arrive at an explicit formula

$$u(x) = \mathbf{E}\left\{\int_0^\infty F(\xi_t^x)\mathrm{e}^{tB}\,\mathrm{d}t\right\}, \quad \xi_t^x = x + t\vec{A} + \Lambda\vec{w}_t, \tag{4.3}$$

which determines the solution of Eq. (4.1) as the mathematical expectation computed over trajectories of the random motion ξ_t^x determined by the standard Brownian motion $\vec{w}_t$ transformed by the diagonal matrix $\Lambda = \mathrm{diag}[\Lambda_{11},\dots,\Lambda_{NN}]$. Similarly, combining (4.2) with the Bismut formulas (3.25)–(3.27) we conclude that the derivatives of the solution of (4.1) are represented by the mathematical expectations

$$D_{\mathfrak{v}}^k u(x) = \mathbf{E}\left\{\int_0^\infty F(\xi_t^x)\mathrm{e}^{tB}\mathscr{D}_{\mathfrak{v}}^k([\xi_t^x])\right\}, \quad \mathfrak{v} = \{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_k\}, \tag{4.4}$$

with the differentiating functionals $\mathscr{D}_{\mathfrak{v}}^k$ from (3.25).

To simplify further the expressions (4.3) and (4.4) we recall the Monte-Carlo quadrature

$$\int_0^\infty F(t)\,\mathrm{d}t = \frac{1}{\beta}\mathbf{E}\{F(\tau)\mathrm{e}^{\beta\tau}\}, \quad \beta>0, \tag{4.5}$$

where the mathematical expectation is computed over the random numbers τ distributed by the well-known (Feller, 1967; Rozanov, 1995) exponential law

$$\mathbf{P}(\tau>t) = \mathrm{e}^{-\beta t}. \tag{4.6}$$

Then, applying (4.5) to (4.3) and to (4.4) we represent the solution $u(x)\equiv D^0u(x)$ of Eq. (4.1) and its derivatives $D_{\mathfrak{v}}^k u(x)$ as the mathematical expectations

$$D_{\mathfrak{v}}^k u(x) = \frac{1}{\beta}\mathbf{E}\{F(\xi_\tau^x)\mathrm{e}^{\tau(B+\beta)}\mathscr{D}_{\mathfrak{v}}^k([\xi_\tau^x])\}, \quad \xi_t^x = x + \Lambda\vec{w}_t, \tag{4.7}$$

with the differentiating functionals $\mathscr{D}_{\mathfrak{v}}^k([\xi_t^x])$ from (3.26), and with the averaging extended over the trajectories of the random motion ξ_t^x from (4.3) launched at the time $t=0$ from x and interrupted at the random instant τ with the exponential distribution (4.6).

Since formulas (4.7) remain valid for a broad class of functionals $\mathscr{D}_{\mathfrak{v}}^k([\xi_t^x])$ described by the integral (3.25), it is not surprising that by restricting these functionals to narrower classes one may obtain simpler representations than (4.7) for the derivatives of the solution of (4.1). For example, the first-order derivative of $u(x)$ along the vector $\vec{v}$ can be represented by the formula

$$D_{\vec{v}}^1 u(x) = \mathbf{E}\left\{\int_0^\infty F(\xi_t^x)\mathrm{e}^{tB}\left(\Lambda^{-1}\vec{v}\cdot\mathrm{d}\vec{w}_t\right)\right\}, \tag{4.8}$$

whose only distinction from the Feynman–Kac formula (4.3) representing $u(x)$ is that the integration by the Lebesgue measure dt appearing in (4.3) is replaced in (4.8) by the integration with respect to the stochastic measure $\left(\Lambda^{-1}\vec{v}\cdot \mathrm{d}\vec{w}_t\right)$. Indeed, to derive (4.8) we apply (4.2) and represent the derivative of $u(x)$ as the sum

$$D^1_{\vec{v}}u(x) = \lim_{\Delta t\to 0}\sum_{k=0}^{\infty} D^1_{\vec{v}}\phi(x,t_k)\Delta t, \quad t_k = k\Delta t. \tag{4.9}$$

Then, we compute the derivatives $D^1_{\vec{v}}\phi(x,t_k)$ by the Bismut formulas

$$D^1_{\vec{v}}\phi(x,t_k) = \mathbf{E}\left\{ f(\xi^x_{t_k})\mathrm{e}^{Bt_k}\frac{(\Lambda^{-1}\vec{v}\cdot\Delta\vec{w}_{t_k})}{\Delta t}\right\}, \quad \Delta\vec{w}_{t_k}\equiv\int_{t_k}^{t_{k+1}}\Lambda^{-1}\vec{v}\cdot\mathrm{d}\vec{w}_t, \tag{4.10}$$

which follow from (3.26) with the density $p_{t_k}(s)$ uniformly distributed on the interval $t_k\leqslant s < t_{k+1}$ and vanishing for any other s. Finally, substituting (4.10) into (4.9) and passing to the limit $\Delta t\to 0$ we arrive at (4.8).

To get an indication of the practical efficiency of the obtained probabilistic solutions of the elliptic equations (4.1) we consider a particular two-dimensional equation

$$\frac{1}{2}(2u''_{xx}+u''_{yy}) + (u'_x + u'_y) - \frac{1}{2}u + F = 0, \tag{4.11}$$

where

$$F(x) = \left(\frac{7}{2} + 2x + 2y - 4x^2 - 2y^2\right)\frac{\mathrm{e}^{-(x^2+y^2)}}{\pi}, \tag{4.12}$$

which has the following exact solution and its derivatives

$$u = \frac{\mathrm{e}^{-(x^2+y^2)}}{\pi}, \quad u'_x = -2xu, \quad u'_y = -2yu, \quad u''_{xy} = 4xyu. \tag{4.13}$$

Fig. 1 shows the results of the numerical simulation of the function $u(x,y)$ from (4.13) considered as the solution of (4.11). The first diagram of Fig. 1 shows the profile of $u(x,y)$ along the horizontal axis $y = 0.2$. The continuous line shows the exact values of $u(x,y)$ computed by (4.12), while the circles '$\circ$' mark $u(x,y)$ simulated by the probabilistic formula (4.7) with $k = 0$. Similarly, continuous lines in the second diagram show the first-order derivatives $u'_x(x,y)$ and $u'_y(x,y)$ computed by the analytic formulas from (4.13), while the '$\triangleright$' and '$\triangle$' mark, respectively, the values of $u'_x(x,y)$ and $u'_y(x,y)$ simulated by the formulas (4.7) with $k = 1$ and with the 'differentiating' functionals

$$\mathscr{D}^1_{\vec{v}}([\xi_t]) = \frac{1}{t}\int_0^t \vec{v}\cdot\mathrm{d}\vec{w}_s \equiv \frac{\vec{v}\cdot\vec{w}_t}{t}, \quad \text{where } \vec{v}=\vec{e}_x, \text{ or } \vec{v}=\vec{e}_y, \tag{4.14}$$

defined by the integral (3.25), with the density $p_t(s)$ uniformly distributed on $0 < s < t$. Finally, the second derivative $u''_{xy}(x,y)$ of the solution of (4.11) is displayed on the third diagram, where the continuous line corresponds to the exact values, and diamonds '$\diamond$' correspond to this derivative simulated by the formula (4.7) with the differentiating functional

$$\mathscr{D}^2_{\vec{e}_x\vec{e}_y}([\xi_t]) = \frac{1}{t^2}\int_0^t\int_0^t(\vec{e}_x\cdot\mathrm{d}\vec{w}_{s_x})(\vec{e}_x\cdot\mathrm{d}\vec{w}_{s_y}) \equiv \frac{(\vec{e}_x\cdot\vec{w}_t)(\vec{e}_y\cdot\vec{w}_t)}{t^2}, \tag{4.15}$$

The mathematical expectations were estimated by averaging over the total number of 6000 independent random walks

$$(x,y)\to(x,y)+\vec{A}\tau+\sqrt{\tau}\vec{w}_\tau \quad \vec{A}=(1,1), \tag{4.16}$$

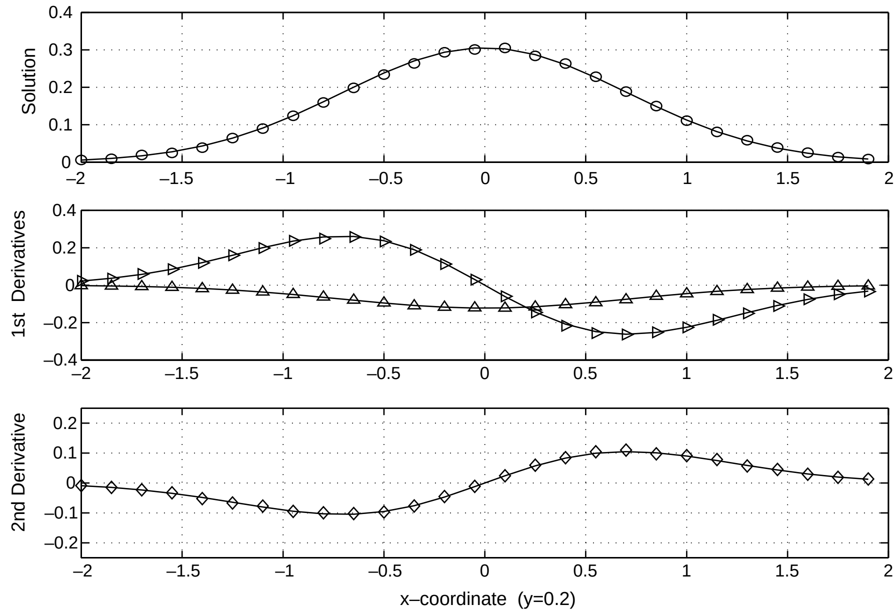

Fig. 1. Simulations by the Feynman–Kac and Bismut formulas.

where $\vec{w}_\tau$ is a random vector normally distributed in $\mathbb{R}^2$, and τ is a random number distributed by the exponential law (4.6) with the parameter $\beta = 1/2$. To reduce the variance of the probabilistic computations we included in the averaging the path (4.16) driven by the Brownian motion $\vec{w}_\tau$, together with three associated paths: two paths corresponding to the Brownian motions symmetrical to $\vec{w}_\tau$ with respect to the Cartesian axes, and one path driven by the Brownian motion symmetrical to $\vec{w}_\tau$ with respect to the center of coordinates.

To conclude this section it is instructive to discuss the Poisson equation

$$\frac{1}{2}\nabla^2 u + F = 0, \tag{4.17}$$

which may be treated as a particular case of (4.1) with $\vec{A} = 0$, $B = 0$ and with $\Lambda_{ii} = 1$ for all $1 \leqslant i \leqslant N$. The difficulty of the probabilistic analysis of this problem comes from the observation (Busnello, 1999; Revuz and Yor, 1991) that the Feynman–Kac formula (4.3) corresponding to Eq. (4.17) considered in the entire space $\mathbb{R}^N$ diverges. However, it is also known (Busnello, 1999) that if $F \in L^p \cap L^q$, with $1 \leqslant p < 2 < q \leqslant \infty$, then the Bismut formula

$$D_{\vec{v}} u(x) = \mathbf{E}\left\{ \int_0^\infty F(x + \vec{w}_t) \frac{\vec{v} \cdot \vec{w}_t}{t} \mathrm{d}t \right\}, \tag{4.18}$$

converges to the derivative of the solution $u(x)$ of (4.17) along the vector $\vec{v}$. Therefore, applying the quadrature (4.5) with some positive parameter $\beta > 0$ we can convert (4.18) to the convergent formula

$$D_{\vec{v}} u(x) = \frac{1}{\beta} \mathbf{E}\left\{ F(x + \vec{w}_\tau) \mathrm{e}^{\beta\tau} \frac{\vec{v} \cdot \vec{w}_\tau}{\tau} \right\}, \quad \beta > 0, \tag{4.19}$$

where the averaging is extended over all Brownian motions $\vec{w}_t$ interrupted at the random time τ distributed by the exponential law (4.6) with the parameter $\beta > 0$. Similarly, for the second-order derivatives of $u(x)$ we obtain the representation

$$D^2_{\vec{v}_1\vec{v}_2}u(x) = \frac{1}{\beta}\mathbf{E}\left\{F(x+\vec{w}_\tau)e^{\beta\tau}\frac{(\vec{v}_1\cdot\vec{w}_\tau)(\vec{v}_2\cdot\vec{w}_\tau)}{\tau^2}\right\}, \quad \beta > 0, \tag{4.20}$$

where all of the notation retains its meaning from (4.19).

It should be emphasized that the Bismut formulas (4.19) and (4.20) representing derivatives of the solution $u(x)$ of the Poisson equation (4.17) converge even though the Feynman–Kac formula (4.3) for $u(x)$ diverges. This difference is due to the additional factors whose estimates

$$\mathbf{E}\left\{\frac{|\vec{v}\cdot\vec{w}_\tau|}{\tau}\right\} = O\left(\frac{1}{\sqrt{\tau}}\right), \quad \tau\to\infty \tag{4.21}$$

guarantee at least slow convergence of (4.18) and, consequently, of (4.19) and (4.20), even when $\mathbf{E}(|F(\xi^x_t)|) = O(1)$, which is not sufficient for the convergence of (4.3) with $B\equiv 0$. Such differences in the convergence of the Bismut and Feynman–Kac formulas is characteristic for equations of the type (4.1) with $\vec{A} = 0$ and $B = 0$. But when $B < 0$, as in the case of (2.23), which arises in the analysis of Dirichlet boundary value problems in a domain G, the convergence is guaranteed both for the Feynman–Kac and Bismut formulas.

To illustrate the divergence and convergence properties of the probabilistic formulas (4.19) and (4.20) for the Poisson equation we applied these formulas to Eq. (4.17) on the plane $\mathbb{R}^2$ with the function

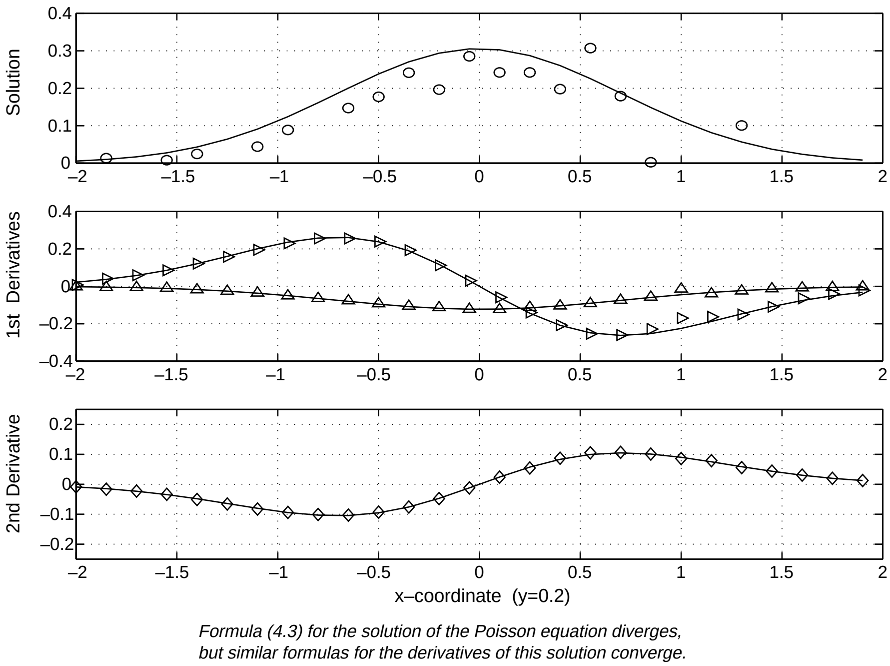

Fig. 2. Probabilistic solution of the Poisson equation.

$$F(x,y) = 2(1 - x^2 - y^2)\frac{e^{-(x^2+y^2)}}{\pi}, \tag{4.22}$$

which implies that

$$u = \frac{e^{-(x^2+y^2)}}{\pi}, \quad u'_x = -2xu, \quad u'_y = -2yu, \quad u''_{xy} = 4xyu. \tag{4.23}$$

The results of the numerical simulation of the functions from (4.23) are presented on Fig. 2 in the same format as used on Fig. 1. The mathematical expectations were estimated by averaging over 10,000 independent random walks (4.16) with $\vec{A} = 0$ and with the parameter β set to $\beta = 0.5$. The first diagram, displaying the function $u(x,y)$ clearly shows divergence of the Feynman–Kac formula (4.3) corresponding to the Poisson equation (4.17) and (4.22). At the same time, the second and the third diagrams of Fig. 2 demonstrate convergence of the formulas (4.19) and (4.20) representing first and second derivatives of $u(x,y)$.

5. Probabilistic solution of a triangular system of equations

To make our approach to systems of partial differential equations more transparent we first consider a triangular system

$$\mathscr{L}_1 u_1 + \gamma D^2_{\vec{v}_1\vec{v}_2} u_2 + F_1 = 0, \tag{5.1}$$

$$\mathscr{L}_2 u_2 + F_2 = 0, \tag{5.2}$$

where γ is a constant, $\vec{v}_1$ and $\vec{v}_2$ are independent constant vectors, F_k are smooth functions and

$$\mathscr{L}_k = \frac{1}{2}\sum_{i=1}^{N} \Lambda^2_{k,ii}\frac{\partial^2}{\partial x_i^2} + \vec{A}_k \cdot \vec{\nabla} + B_k, \quad k = 1, 2, \tag{5.3}$$

are second-order differential operators with constant coefficients.

Eqs. (5.1) and (5.2) may certainly be solved by two subsequent applications of formulas of the type (4.7). Indeed, we may first compute u_2 by the formula (4.7) applied to Eq. (5.2). Then, applying (4.7) to Eq. $\mathscr{L}_1 u_2 + F = 0$ with the already defined function $F = F_1 + \gamma D^2_{\vec{v}_1\vec{v}_2} u_2$, we compute u_1. It is clear that the straightforward implementation of this iterative approach requires computation of u_2 over the entire space, which eliminates one of the main advantages of probabilistic solutions, that is, the possibility of computing the unknown functions only at certain required points. However, we will see here that the use of probabilistic formulas makes it possible to implement the described iterations in a direct one-step procedure which efficiently solves triangular systems of equations of the type (5.1) and admits generalization to more general systems of differential equations, such as the Lamé equations of the theory of elasticity discussed in Section 6.

Assume for the moment that u_2 is already known. Then (5.1) can be treated as an equation of the type (4.1) with respect to the unknown function u_1 and with the pre-defined term $F = F_1 + \gamma D^2_{\vec{v}_1\vec{v}_2} u_2$. Therefore, after introducing the diagonal matrices

$$\Lambda_k = \mathrm{diag}[\Lambda_{k,11}, \Lambda_{k,22}, \ldots, \Lambda_{k,NN}], \quad k = 1, 2, \tag{5.4}$$

and applying (4.7) we arrive at the identity

$$u_1(x) = \frac{1}{\beta}\mathbf{E}\left\{F(\xi^x_{\tau_1})e^{\tau_1(B_1+\beta)} + \gamma\left[D^2_{\vec{v}_1\vec{v}_2} u_2(\xi^x_{\tau_1})\right]e^{\tau_1(B_1+\beta)}\right\}, \tag{5.5}$$

where the averaging is extended over the trajectories of the random motion

$$\xi_t^x = x + t\vec{A}_1 + \Lambda_1\vec{w}_t, \quad 0 \leqslant t < \tau_1, \tag{5.6}$$

launched at the moment $t = 0$ from the point x, and interrupted at the random time τ_1 with the exponential distribution (4.6).

Then, applying the Bismut formula (4.7) to (5.2) we represent the derivative $D_{\mathfrak{v}}u_2(\xi_{\tau_1}^x)$ by the formula

$$D_{\mathfrak{v}}u_2(\xi_{\tau_1}^x) = \mathbf{E}\Big\{F_2(\xi_{\tau_2}^x)\mathrm{e}^{(\tau_2-\tau_1)(B_2+\beta)}Q\Big\}, \tag{5.7}$$

where

$$Q = \frac{[\vec{v}_1 \cdot \Lambda_1(\vec{w}_{\tau_2} - \vec{w}_{\tau_1})][\vec{v}_2 \cdot \Lambda_2(\vec{w}_{\tau_2} - \vec{w}_{\tau_1})]}{\beta\Lambda_{2,11}\Lambda_{2,22}(\tau_2 - \tau_1)^2}, \tag{5.8}$$

and the mathematical expectation is computed over the trajectories of the Brownian motion

$$\xi_t^x = \xi_{\tau_1}^x + (t - \tau_1)\vec{A} + \Lambda_2\vec{w}_{t-\tau_1}, \quad \tau_1 \leqslant t < \tau_2, \tag{5.9}$$

passing at the instant $t = \tau_1$ through the point $x_1 = \xi_{\tau_1}^x$, and interrupted at the random time τ_2 distributed by the exponential law

$$\mathbf{P}(\tau_2 - \tau_1 > t) = \mathrm{e}^{-\beta(\tau_2-\tau_1)}. \tag{5.10}$$

Finally, combining formulas (5.5) and (5.7) we arrive at the expression

$$u_1(x) = \mathbf{E}\Big\{F_1(\xi_{\tau_1}^x)\mathrm{e}^{\tau_1(B_1+\beta)} + F_2(\xi_{\tau_2}^x)Q\mathrm{e}^{\tau_1(B_1+\beta)+\tau_2(B_2+\beta)}\Big\}, \tag{5.11}$$

which directly represents the solution u_1 of the system (5.1) and (5.2) by the mathematical expectation computed over the two random numbers τ_1, and τ_2 distributed according to (4.6) and (5.10), and over the trajectories of the random motion launched from the observation point x and controlled by the stochastic equations (5.6) and (5.9). Similarly, for the derivatives $D_{\mathfrak{v}}^k u_1$ of u_1 along the k-tuples $\mathfrak{v} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ we get the expression

$$D_{\mathfrak{v}}^k u_1(x) = \mathbf{E}\Big\{\mathscr{D}_{\mathfrak{v}}^k([\xi_{\tau_1}^x])\Big[F_1(\xi_{\tau_1}^x)\mathrm{e}^{\tau_1(B_1+\beta)} + F_2(\xi_{\tau_2}^x)Q\mathrm{e}^{\tau_1(B_1+\beta)+\tau_2(B_2+\beta)}\Big]\Big\}, \tag{5.12}$$

whose only difference with (5.11) is the presence of the differentiating functional $\mathscr{D}_{\mathfrak{v}}^k([\xi_{\tau_1}^x])$ defined by formulas of the type (3.26) and (3.27), and depending on the trajectory of the random motion ξ_t^x on the initial time-interval $0 \leqslant t < \tau_1$. As for the function u_2 and its derivatives, they can be computed by the Feynman–Kac–Bismut formulas (4.7) applied to Eq. (5.2).

6. Probabilistic solution of Lamé-like equations

Finally we consider a system of elliptic second-order differential equations

$$\mathscr{L}_1u_1 + \gamma D_{\mathfrak{e}}u_2 + F_1 = 0, \quad D_{\mathfrak{e}} \equiv D_{\vec{e}_1\vec{e}_2}, \tag{6.1}$$

$$\mathscr{L}_2u_2 + \gamma D_{\mathfrak{e}}u_1 + F_2 = 0, \tag{6.2}$$

where $\vec{e}_1$, $\vec{e}_2$ and γ are constants, $F_1(x)$ and $F_2(x)$ are bounded functions of $x \in \mathbb{R}^N$,

$$\mathscr{L}_k = \frac{1}{2}\sum_{i=1}^{N}\Lambda_{k,ii}^2\frac{\partial^2}{\partial x_i^2} + \vec{A}_k \cdot \vec{\nabla} + B_k, \quad k = 1, 2, \tag{6.3}$$

are scalar differential operators with constant coefficients. It is clear that the Lamé equations (Sokolnikoff, 1956) describing the plain-strain state of an elastic medium can be viewed as a two-dimensional case of (6.1) and (6.2) characterized by the coefficients

$$\begin{aligned} &\Lambda^2_{1,11} = \Lambda^2_{2,22} = \lambda + 2\mu, \quad \Lambda^2_{1,22} = \Lambda^2_{2,11} = \mu, \quad \gamma = \lambda + \mu, \\ &\vec{A}_1 = \vec{A}_2 = 0, \quad B_1 = B_2 = 0, \end{aligned} \tag{6.4}$$

where λ and μ are the material constants of the elastic medium. Here we develop a probabilistic approach to the system of equations (6.1) and (6.2).

To analyze this system by the iterative scheme applied above to a triangular system (5.1), we first introduce diagonal matrices

$$\Lambda_1 = \mathrm{diag}[\Lambda_{1,11}, \Lambda_{1,22}, \ldots, \Lambda_{1,NN}], \quad \Lambda_2 = \mathrm{diag}[\Lambda_{2,11}, \Lambda_{2,22}, \ldots, \Lambda_{2,NN}] \tag{6.5}$$

generated by the coefficients of Eqs. (6.1) and (6.2) respectively. Then, treating (6.1) as an equation of the type (4.1) with respect to u_1, we derive from (4.7) the identity

$$u_1(x) = \mathbf{E}\Big\{Q_1 F(\xi^x_{\tau_1}) + \gamma Q_1 D_e u_2(\xi^x_{\tau_1})\Big\}, \quad Q_1 = \mathrm{e}^{(B_1+\beta)\tau_1}/\beta, \tag{6.6}$$

where the mathematical expectation is computed over the trajectories of the random motion ξ^x_t, which starts at the moment $t = 0$ from x, runs thereafter as

$$\xi^x_t = x + t\vec{A}_1 + \Lambda_1 \vec{w}_t, \quad 0 \leqslant t < \tau_1, \tag{6.7}$$

where $\vec{w}_t$ is the standard N-dimensional Brownian motion, and stops at the random time τ_1 distributed by the exponential law $\mathbf{P}(\tau_1 > t) = \mathrm{e}^{-\beta t}$ with some positive parameter $\beta > 0$.

After that, we treat (6.2) as an equation with respect to u_2 and applying (4.7), represent the derivative $D_e u_2(\xi^x_{\tau_1})$ appearing in the right-hand side of (6.6) by the formula

$$D_e u_2(\xi^x_{\tau_1}) = \mathbf{E}\Big\{qF_2\Big(\xi^x_{\tau_2}\Big) + \gamma q D_e u_1\Big(\xi^x_{\tau_2}\Big)\Big\}, \tag{6.8}$$

with the factor

$$q = \mathscr{D}_e([\xi^x_t])\mathrm{e}^{(B_2+\beta)\tau_2}/\beta, \quad \tau_1 \leqslant t < \tau_2, \tag{6.9}$$

where $\mathscr{D}_e([\xi^x_t])$ is a functional (3.26) and (3.27), defined on trajectories of the random motion

$$\xi^x_t = \xi^x_{\tau_1} + (t - \tau_1)\vec{A}_2 + \Lambda_2 \vec{w}_{t-\tau_1}, \quad \tau_1 \leqslant t < \tau_2, \tag{6.10}$$

running on the time interval $\tau_1 \leqslant t < \tau_2$, where τ_2 is a random number distributed by the law $\mathbf{P}(\tau_2 - \tau_1 > t) = \mathrm{e}^{-\beta(\tau_2-\tau_2)}$ with the same parameter $\beta > 0$ as in (6.6).

As a result, after the substitution of (6.8) into (6.6) we obtain the identity

$$u_1(x) = \mathbf{E}\Big\{Q_1 F_1(\xi^x_{\tau_1}) + \gamma q Q_1 F_2(\xi^x_{\tau_2}) + \gamma^2 q Q_1 D_e u_1(\xi^x_{\tau_2})\Big\}, \tag{6.11}$$

whose right-hand side contains the derivative $D_e u_1(\xi^x_{\tau_2})$, which can be represented by the Bismut formula (4.7) applied to Eq. (6.1), etc.

It is clear that the described iterations can be started from Eq. (6.2) as well as from (6.1), and that these iterations can be continued indefinitely resulting in the series

$$u_n(x) = \mathbf{E}\left\{\sum_{k=1}^{\infty} Q_k F_{\nu^n_k}(\xi^{x,n}_{\tau_k})\right\}, \quad n = 1, 2, \tag{6.12}$$

whose components have the meaning described below.

The indices v_k^n take one of the two possible values $v = 1$ or $v = 2$ determined by the rule

$$v_k^n = \begin{cases} n, & \text{if } k = 1, 3, 5, \ldots, \\ \bar{n}, & \text{if } k = 2, 4, 6, \ldots, \end{cases} \tag{6.13}$$

where

$$\bar{n} = \begin{cases} 1, & \text{if } n = 2, \\ 2, & \text{if } n = 1. \end{cases} \tag{6.14}$$

The random numbers τ_k form a monotonic sequence

$$0 \equiv \tau_0 < \tau_1 < \tau_2 \cdots < \tau_{k-1} < \tau_k < \cdots, \tag{6.15}$$

known as the Poissonian stream (Feller, 1967; Rozanov, 1995) characterized by the exponential distributions

$$\mathbf{P}(\tau_k - \tau_{k-1} > t) = e^{-\beta t}. \tag{6.16}$$

The continuous random motions $\xi_t^{x,n}$ are launched at the time $t = 0$ from the observation point x and are controlled thereafter by the stochastic equation

$$\xi_0^{x,n} = x, \quad d\xi_t^{x,n} = \begin{cases} \Lambda_n d\vec{w}_t, & \text{if } \tau_{2j} \leqslant t < \tau_{2j+1}, \\ \Lambda_{\bar{n}} d\vec{w}_t, & \text{if } \tau_{2j-1} \leqslant t < \tau_{2j}. \end{cases} \tag{6.17}$$

The factors Q_k are defined by the recursive formulas

$$Q_1^n = e^{(B_n+\beta)\tau_1}/\beta, \tag{6.18}$$

$$Q_k = Q_{k-1}\mathscr{D}_e([\xi_t^{x,n}])e^{(B_v+\beta)(\tau_v-\tau_{v-1})}/\beta, \qquad v = v_k^n, \quad \tau_{k-1} \leqslant t < \tau_k, \tag{6.19}$$

involving the values of the functional $\mathscr{D}_e([\xi_t^{x,n}])$ defined on the trajectories of the random motion $\xi_t^{x,n}$ between consequent moments τ_k of the Poissonian stream (6.15).

It is worth mentioning that formulas (6.12) for the solutions of Eqs. (6.1) and (6.2) can be readily modified to represent also the derivatives of these solutions. Thus, for the derivatives $D_{\mathfrak{v}}^m u_n$ of u_n along the m-tuples $\mathfrak{v} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ we get the expression

$$D_{\mathfrak{v}}^m u_1 = n(x) = \mathbf{E}\left\{\mathscr{D}_{\mathfrak{v}}^m([\xi_{\tau_1}^{x,1}])\sum_{k=1}^{\infty} Q_k F_{v_k^n}(\xi_{\tau_k}^{x,n})\right\}, \quad n = 1, 2, \tag{6.20}$$

whose only difference with (6.12) is the presence of the differentiating functional $\mathscr{D}_{\mathfrak{v}}^m([\xi_{\tau_1}^x])$ defined by the formulas of the type (3.26) and (3.27), and depending on the trajectory of the random motion $\xi_t^{x,1}$ on the initial time-interval $0 \leqslant t < \tau_1$.

To discuss the convergence of the series (6.12) it is convenient to look at it from another point of view. Let $\mathscr{L}_k^{-1}$ be the operators inverse to $\mathscr{L}_k$ from (6.3), existing due to the ellipticity of $\mathscr{L}_k$. Then, Eqs. (6.1) and (6.2) can be converted to the form

$$\begin{aligned} u_1 + \gamma\mathscr{L}_1^{-1}D_e u_2 + \mathscr{L}_1^{-1}F_1 = 0, \\ u_2 + \gamma\mathscr{L}_2^{-1}D_e u_1 + \mathscr{L}_2^{-1}F_2 = 0, \end{aligned} \tag{6.21}$$

which can be split, after one iteration, to a system of two uncoupled equations

$$u_1 - T_1 u_1 = f_1, \quad u_2 - T_2 u_2 = f_2, \tag{6.22}$$

with the operators

$$T_1 = \gamma^2 \mathscr{L}_1^{-1} D_e \mathscr{L}_2^{-1} D_e, \quad T_2 = \gamma^2 \mathscr{L}_2^{-1} D_e \mathscr{L}_1^{-1} D_e, \tag{6.23}$$

and with the right-hand sides

$$f_1 = \gamma \mathscr{L}_1^{-1} D_e \mathscr{L}_2^{-1} F_2 - \mathscr{L}_1^{-1} F_1, \quad f_2 = \gamma \mathscr{L}_2^{-1} D_e \mathscr{L}_1^{-1} F_1 - \mathscr{L}_2^{-1} F_2. \tag{6.24}$$

Eq. (6.22) can be formally resolved by the Neumann series

$$u_n = f_n + T_n f_n + T_n^2 f_n + T_n^3 f_n + \cdots, \quad n = 1, 2, \tag{6.25}$$

and comparing the procedures that led to (6.25) and to (6.12), one can conclude that the probabilistic formulas (6.12) are just special implementations of (6.25) in a similar way that the probabilistic solution (5.11) of the triangular system (5.1) is just a special implementation of the solution of (5.1) obtained by the obvious two-step iterative process outlined in the beginning of Section 5.

From the above it follows that the probabilistic series (6.12) converges to the solution of (6.1) and (6.2) when and only when the Neumann series (6.25) converge. Therefore, to justify the probabilistic approach to the plain-strain problems of the theory of elasticity it suffices to prove convergence of the Neumann series (6.25) corresponding to the two-dimensional Lamé equations which have the structure (6.1)–(6.3) with the coefficients from (6.4).

Let $\hat{u}_n(w_1, w_2)$, where $n = 1, 2$, be the Fourier transforms

$$\hat{u}_n(w_1, w_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u_n(x_1, x_2) e^{i(x_1 w_1 + x_2 w_2)} \, dx_1 \, dx_2, \tag{6.26}$$

of the unknown functions of the Lamé equations (6.1)–(6.4). Then, Eq. (6.22) associated with (6.1)–(6.4) are converted by the Fourier transform to the algebraic equations

$$\left(1 - \widehat{T}\right) \hat{u}_n = \hat{f}_n, \quad n = 1, 2, \tag{6.27}$$

where $\hat{f}_n$ are the Fourier transforms of the right-hand side functions f_n from (6.24), and

$$\widehat{T} \equiv \widehat{T}(w_1, w_2) = \frac{(\lambda + \mu)^2 w_1^2 w_2^2}{[(\lambda + 2\mu) w_1^2 + \mu w_2^2][(\lambda + 2\mu) w_2^2 + \mu w_1^2]}, \tag{6.28}$$

as follows from the definitions (6.24) of the operators T_n involved in (6.22). Finally, we observe that the obvious inequalities

$$(\lambda + 2\mu) w_1^2 + \mu w_2^2 \geqslant (\lambda + 2\mu) w_1^2, \quad (\lambda + 2\mu) w_2^2 + \mu w_1^2 \geqslant (\lambda + 2\mu) w_2^2, \tag{6.29}$$

generate the estimate

$$|\widehat{T}| \leqslant \frac{(\lambda + \mu)^2}{(\lambda + 2\mu)^2} \leqslant 1, \tag{6.30}$$

which guarantees the convergence of the Neumann series for the algebraic equation (6.27) and, consequently, of the Neumann series (6.25), as well as of the probabilistic series (6.12), corresponding to the Lamé equations (6.1)–(6.4).

It is rather clear that the above presented proof of convergence of the series (6.12) and (6.25) corresponding to the two-dimensional Lamé equations can be extended to general elliptic systems of the type (6.1) and (6.2), but here we do not go into the details of such an analysis.

To get an indication of the practicability of the probabilistic approach to systems of differential equations we considered the following system of two-dimensional equations:

$$\frac{1}{2}\left(u''_{xx} + \tfrac{1}{3} u''_{yy}\right) - u + \frac{1}{3} v''_{xy} = f_1, \quad \frac{1}{2}\left(\tfrac{1}{3} v''_{xx} + v''_{yy}\right) - v + \frac{1}{3} u''_{xy} = f_2, \tag{6.31}$$

with the factors $\frac{1}{2}$ introduced to make (6.31) match the structure in (6.1)–(6.4) with the operators

$$\mathscr{L}_1 = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{1}{3}\frac{\partial^2}{\partial y^2}\right), \quad \mathscr{L}_2 = \frac{1}{2}\left(\frac{1}{3}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \quad \gamma D_e = \frac{1}{3}\frac{\partial^2}{\partial x \partial y}, \tag{6.32}$$

that arise in the analysis of the plain-strain state of the elastic material with Poisson's ratio $\sigma = 0.25$, typical for steels. The right-hand sides f_1 and f_2 in (6.31) are defined by the formulas

$$\begin{aligned} f_1(x,y) &= \mathscr{L}_1\chi\left(x,y;\tfrac{1}{2},1\right) - \chi\left(x,y;\tfrac{1}{2},1\right) + \gamma D_e\chi\left(x,y;1,\tfrac{1}{4}\right), \\ f_2(x,y) &= \mathscr{L}_2\chi\left(x,y;1,\tfrac{1}{4}\right) - \chi\left(x,y;1,\tfrac{1}{4}\right) + \gamma D_e\chi\left(x,y;\tfrac{1}{2},1\right), \end{aligned} \tag{6.33}$$

where

$$\chi(x,y;D,p) = \frac{p\mathrm{e}^{-(x^2+y^2)/2D}}{2\pi D}, \tag{6.34}$$

is the scaled Gaussian function.

It is clear that the exact solution of Eqs. (6.31) and (6.33) is

$$u(x) = \chi\left(x,y;\tfrac{1}{2},1\right), \quad v(x) = \chi\left(x,y;1,\tfrac{1}{4}\right). \tag{6.35}$$

On the other hand, these equations can be considered as a particular case of the system of equations (6.1) and (6.2) with the parameters

$$\begin{aligned} &\Lambda_{1,11} = 1, \quad \Lambda_{1,22} = \frac{1}{\sqrt{3}}, \quad B_1 = -1, \quad \gamma = \frac{1}{3}, \\ &\Lambda_{2,11} = \frac{1}{\sqrt{3}}, \quad \Lambda_{2,22} = 1, \quad B_2 = -1, \quad \vec{e}_1 = \vec{e}_x, \quad \vec{e}_2 = \vec{e}_y, \end{aligned} \tag{6.36}$$

where $\vec{e}_x$ and $\vec{e}_y$ are the Cartesian coordinate vectors.

The first diagram of Fig. 3 shows the solution $u(x,y)$ along the line $y = 0.2$. The second diagram shows its first derivatives and the third diagram shows the mixed second-order derivative $u''_{xy}(x,y)$. The continuous lines in all diagrams display the results computed by the analytic formulas (6.35), while the circles '$\circ$' and diamonds '$\diamondsuit$' mark u and u''_{xy}, respectively, simulated by the probabilistic formulas (6.12)–(6.20). Similarly, in the second diagram the triangles '$\triangleright$' and '$\triangle$' mark the simulated values of u'_x and u'_y, respectively. All mathematical expectations were approximated by averaging over 10,000 random walks described by Eq. (6.17) and switching modes at the random moments $\tau_1, \tau_2, \ldots, \tau_k$, forming the Poissonian stream (6.15) and (6.16) with the parameter $\beta = \frac{1}{2}$. The numerical results were obtained by retaining five terms in the series (6.12), although the contribution of the 5th term was already practically negligible. To reduce the variance of the probabilistic computations we included in the averaging every random path ξ_t^x driven by the Brownian motion w_t together with the associated paths driven by the Brownian motions symmetrical to w_t with respect to each of the coordinate axes and with respect to the center of the coordinates.

As an another example we consider the system of equations

$$\frac{1}{2}\left(u''_{xx} + \tfrac{1}{3}u''_{yy}\right) + \frac{1}{3}v''_{xy} = f_1, \quad \frac{1}{2}\left(\tfrac{1}{3}v''_{xx} + v''_{yy}\right) + \frac{1}{3}u''_{xy} = f_2, \tag{6.37}$$

with the right-hand sides

$$\begin{aligned} f_1(x,y) &= \mathscr{L}_1\chi\left(x,y;\tfrac{1}{2},1\right) + \gamma D_e\chi\left(x,y;1,\tfrac{1}{4}\right), \\ f_2(x,y) &= \mathscr{L}_2\chi\left(x,y;1,\tfrac{1}{4}\right) + \gamma D_e\chi\left(x,y;\tfrac{1}{2},1\right), \end{aligned} \tag{6.38}$$

defined by the Gaussian function $\chi(x,y;D,p)$ from (6.32) modified by the operators $\mathscr{L}_1$, $\mathscr{L}_2$ and γD_e introduced by (6.32). It is clear that (6.37) are the Lamé equations of the plain-strain state of a medium with

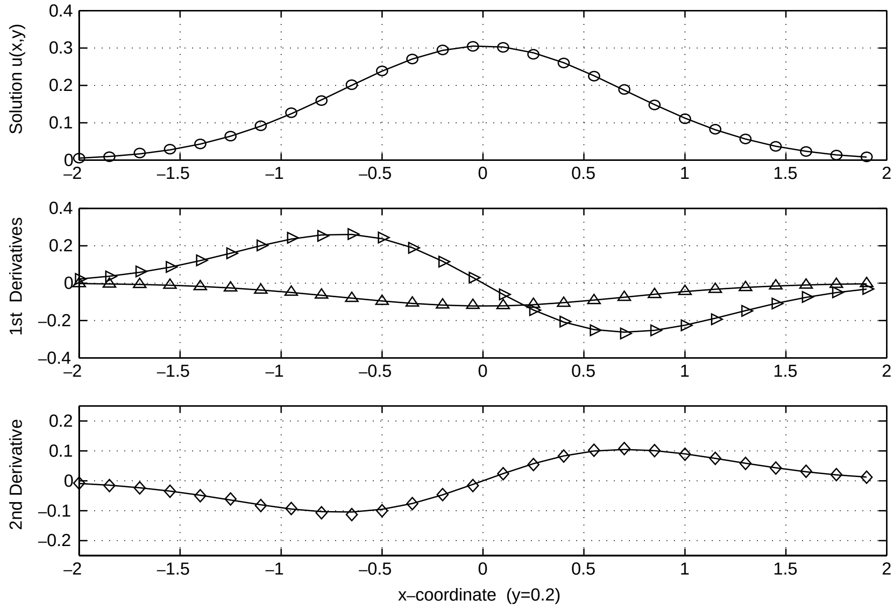

Fig. 3. Probabilistic solution of the system of equations.

the Poisson ratio $\sigma = 0.25$, and that Eq. (6.37) with the right-hand sides (6.38) have the same exact solution (6.35) as in the previous example.

The difference between Eqs. (6.38) and (6.31) is the absence in (6.37) of the damping terms '$-u$' and '$-v$' whose presence guarantees convergence of the expression (6.12) for any bounded functions $f_1(x)$ and $f_2(x)$. From the discussion in the end of Section 4 it follows that the Feynman–Kac formula (6.6) representing u_1 as the solution of (6.1) with pre-defined f_1 and u_2 diverge, from which it follows that the expression (6.12) based on (6.6) also diverges. However, from that discussion it also follows that the Bismut formulas (6.8) and (6.20) still converge to the derivatives of the solutions u and v of the system (6.37). The last observation means that if u and v are Cartesian displacements of the elastic medium in the plain-strain state, then the corresponding components

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{6.39}$$

of the strain tensor can be computed by the probabilistic simulation.

The results of the probabilistic simulation of the solution of Eq. (6.37) are shown in Fig. 4, where the first diagram displays the first-order derivatives u'_x and u'_y, while the second diagram shows the second-order derivative u''_{xy}. All of the computations were made by the same algorithms as used in the previous example.

Here we have applied the random walks approach to systems of partial differential equations with constant coefficients considered in the entire N-dimensional space. Since the results look promising the question arises about possibilities of extending the approach to systems of equations with variable coefficients and to boundary value problems for systems of equations considered in a bounded domain. In this regards it is appropriate to observe that our approach is based on the ability to obtain probabilistic representations of the derivatives of the solution of scalar differential equations. At the end of Section 3 it was mentioned that the Bismut formulas provide such representations and are extendable to problems with

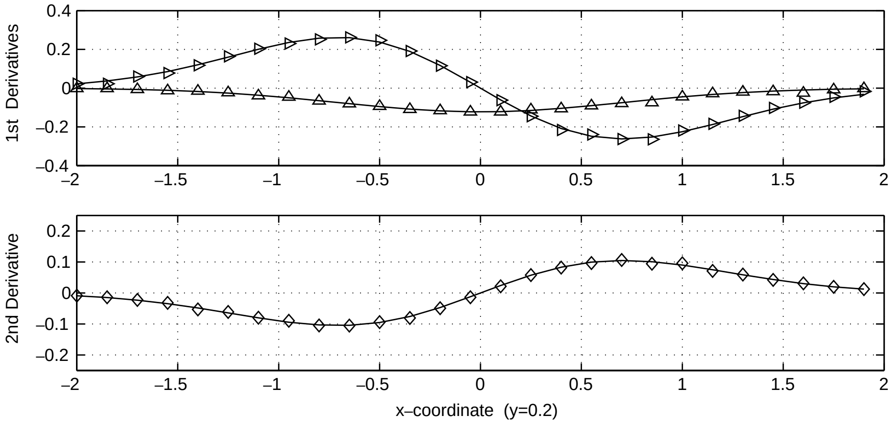

Fig. 4. Probabilistic solution of the system of Lamé equations.

variable coefficients. This indicates that our approach can be extended to systems of equations with variable coefficients and, consequently, as discussed in Section 4, to boundary value problems.

7. Conclusion

The results presented here suggest that the synthesis of the Feynman–Kac and Bismut formulas provides a promising probabilistic approach to partial differential equations of the theory of elasticity. The advantages of this approach include, but are not limited to, versatility, the possibility of computing the functions of interest at isolated points without computing them on massive meshes, and the opportunity of having simple scalable implementations with practically unlimited capability for parallel processing.

Here we explored the basic ideas of the random walk approach to problems of elasticity and considered only elementary examples that illustrate its applications to the analysis of the system of Lamé equations in the entire homogeneous isotropic elastic medium. In future papers we plan to extend the approach to problems of elastostatics in bounded, possibly inhomogeneous and anisotropic media, and also to problems of propagation and diffraction of elastic waves.

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